Towards Formalization of Abstract Simplicial Complexes in Cubical Agda

Willy Ho Lok Chan

willyhlchan@gmail.com

Abstract

This work aims to move the formalization of discrete differential geometry forward in Cubical Agda, following the framework introduced by Langmead [2]. In particular, we give a complete definition of abstract simplicial complexes. Using elements of this type as data, we formalize the geometric realization procedure up to dimension 2. Finally, we formalize the equivalence between a realized n-gon in dimension 1 and the synthetic circle $\mathbb{S}^1$, and provide partial results toward proving that a realized tetrahedron in dimension 2 is equivalent to $\mathbb{S}^2$.

Background Homotopy Type Theory (HoTT) [5] extends Martin-Löf type theory to serve as a foundation for synthetic mathematics. A central feature is Higher Inductive Types (HITs), which construct spaces by directly postulating points and higher paths between them. This facilitates synthetic homotopy theory, where classical topological theorems can be derived easily and checked by proof assistants.

In this spirit, Langmead [2] introduces differential geometry into HoTT by modeling discrete manifolds as abstract simplicial complexes and constructing their homotopical realizations via successive pushouts. The thesis formulates curvature and vector field indices within this setting to establish Gauss–Bonnet and Poincaré–Hopf theorems.[1]

This work formalize most of the content in section 3 of Langmead [2], including abstract simplicial complexes the realization procedure by a seqeunce of pushouts, and equivalence proofs for 1-dimensional complexes. Furthermore, we present partial results toward formalizing the equivalence of the 2-sphere realization left as future work in the original thesis. The code is available at .

Abstract Simplicial Complex An **abstract simplicial complex** M of **dimension** n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \dots, M_n]$ satisfying:

- M_0 is a set of vertices.
- Each M_k is a set of k-**faces**, represented as subsets of M_0 of cardinality $k+1$.
- **Downward closure**: if $F \in M_k$, $G \subseteq F$, and $|G| = j+1$, then $G \in M_j$.

We denote the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \dots, M_k]$ as the k-**skeleton** of M. This structure generalizes meshes and graphs to higher dimensions; for instance, a 1-dimensional complex corresponds directly to a simple graph.

To formalize this in Agda, we first define the type of k-element subsets of an n-element set (appendix A.1). We then define a `Pseudo-ASC` type representing stratified lists

[1] It should be noted that the thesis proves a theorem directly relating total curvature and total index, rather than establishing separate comparisons to a pre-existing synthetic Euler characteristic.

of subsets (appendix A.2). The remaining conditions, namely downward closure and no duplications, are decidable properties. We implement these checks as functions and define `ASC` as a record type bundling the stratified data with two proof-relevant witnesses:

```
record ASC (n k : ℕ) : Type where
  field
    S      : Pseudo-ASC n k
    closed : IsClosed S
    nodup  : NoDup S
open ASC public
```

Realization Procedure The realization of an abstract simplicial complex is defined inductively by dimension. The realization $\mathbb{M}_0$ of a 0-dimensional complex is simply the discrete set M_0, while subsequent dimensions are defined via pushouts:

$$\begin{array}{ccc} M_n \times \partial\Delta(n) & \xrightarrow{\mathrm{pr}_1} & M_n \\ {\scriptstyle \mathbb{A}_n}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle *_{\mathbb{M}_n}} \\ \mathbb{M}_{\leq n-1} & \xrightarrow{\quad\lrcorner\quad} & \mathbb{M}_n \end{array}$$

Here, $\partial\Delta(n)$ denotes the realized $(n-1)$-skeleton of the standard n-simplex $\Delta(n)$. The main formalization challenge lies in the attachment map $\mathbb{A}_n$, which maps the boundary components of each k-face into the realized $(k-1)$-skeleton (see the definition of $\mathbb{A}_1$ in appendix A.3). Because the domain boundary is itself an inductively realized complex, higher coherence constraints make $\mathbb{A}_n$ non-trivial to define for arbitrary n in Agda [4]. We therefore restrict the realization up to dimension 2, for which it suffices to prove the associativity of the `_^_` operator in appendix A.3.

Equivalence of Realized Complexes we establish topological equivalences between realized complexes and standard spheres (e.g., showing that any realized n-gon is equivalent to $\mathbb{S}^1$, and convex polyhedra are equivalent to $\mathbb{S}^2$ .etc). We formalized the 1-dimensional equivalence `n-gon` $\simeq \mathbb{S}^1$ by proving `n-gon` $\simeq$ `(n-1)-gon` and proceeding by induction on n.

Because the original definition for general complexes is not easy to do induction with, we introduce an auxiliary pushout type `CYC` n (appendix A.4). We perform the induction on `CYC`, establishing the isomorphism between `CYC` n and `n-gon` via `pushoutIso`. This structure provides a blueprint for the 2-dimensional sphere equivalence: while defining a tetrahedron complex is straightforward in our framework, completing the full `tetrahedron` $\simeq \mathbb{S}^2$ equivalence remains work in progress, likely requiring an intermediate cell-decomposition pushout.

Remarks & Future Work Immediate next steps include completing the `tetrahedron` $\simeq \mathbb{S}^2$ equivalence proof. To achieve arbitrary n-dimensional realizations, one could explore two-level type theory [1] to manage coherences, establishing equivalences between combinatorial n-spheres and $\mathbb{S}^n$.

Additionally, comparing this formulation with the synthetic CW-complex framework developed by Ljungström and Pujet [3] offers an interesting direction. In particular, it remains an open question whether defining abstract simplicial complexes via CW-complexes with restricted attaching maps provides better computational or proof-ergonomic properties than our model.

References

[1] Altenkirch, T., Capriotti, P., and Kraus, N. (2016). Extending homotopy type theory with strict equality. *CoRR*, abs/1604.03799.

[2] Langmead, G. (2025). Discrete differential geometry in homotopy type theory.

[3] Ljungström, A. and Pujet, L. (2026). Cellular Methods in Homotopy Type Theory. In Faggian, C. and Katoen, J.-P., editors, *41st Annual Symposium on Logic in Computer Science (LICS 2026)*, volume 380 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 66:1–66:25, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[4] Sato, G. (2018). Geometric realization of truncated semi-simplicial sets meta-constructed within HoTT. In *Workshop on Homotopy Type Theory and Univalent Foundations (HoTT/UF 2018)*, Oxford, UK. Extended abstract.

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A Agda Code

A.1 Definition of Subset

```
data § : (n k : ℕ) → Type where
  [] : § zero zero
  ○_ : ∀ {n k} → § n k → § (suc n) k
  ●_ : ∀ {n k} → § n k → § (suc n) (suc k)
```

A.2 Definition of Pseudo-ASC

```
data Pseudo-ASC (n : ℕ) : ℕ → Type where
  ε : Pseudo-ASC n zero
  _∔_ : {k : ℕ} → § n (suc k) → Pseudo-ASC n k → Pseudo-ASC n k
  _#_ : {k : ℕ} → § n (suc (suc k)) → Pseudo-ASC n k → Pseudo-ASC n (suc k)
```

A.3 Implementation of Attachment on Subsets

```
_^_ : {n a b : ℕ} → § n a → § a b → § n b
[] ^ [] = []
○ x ^ y = ○ (x ^ y)
● x ^ ○ y = ○ (x ^ y)
● x ^ ● y = ● (x ^ y)
```

A.4 Simplified Pushout: CYC

```
Cyc : (N : ℕ) → Type
Cyc N = Pushout {A = § N 1 × Bool} {B = § N 1} {C = § N 1} fst att
```